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DIFFERENCE METHODS OF SOLVING THREE-DIMENSIONAL PROBLEMS IN GAS DYNAMICS

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DIFFERENCE METHODS OF SOLVING THREE-DIMENSIONAL PROBLEMS IN GAS DYNAMICS

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ABSTRACT. The problem of stationary flow about an arbitrary smooth body by a supersonic stream of non-viscous non-thermoconductive gas is examined. It is assumed that the body is stationary, and the stream far in front of the body is uniform. The study is restricted to flow in the region between the bow wave and the body up to the next discontinuity.

Let us first of all discuss certain ideas which underlie the compilation of numerical algorithms for solving a problem in three-dimensional flow. /247*

A basic characteristic of the numerical algorithm is its efficiency.

Efficiency in algorithm operation may be described by the number of elementary arithmetic and logic operations necessary to derive a solution of the specified accuracy -- i.e., roughly speaking, it may be described by machine operational time required to solve a problem with the specified accuracy.

Since the degree of maximum error must monotonically decrease as the interval of the difference network becomes smaller in an intelligent numerical algorithm, the number of elementary operations N as a function of the specified error ε comprises a rough estimate of operational efficiency. The economy or efficiency concept is an asymptotic one, i.e., it is important to us how $N(\varepsilon)$ increases when $\varepsilon \to 0$. It is incorrect to describe the algorithm by the quantity $N(\varepsilon)$ for a given value of ε, as is evident from the most elementary examples. Thus, in a problem involving a search for function minima, the functions of one or two variables may be directly factored, but this factoring cannot be done for functions of a large number of variables.

Another illustration which is closer to us is the boundary value problem for a system of nonlinear differential equations. Fitting is a more or less unique algorithm for solving such a system. However, if this fitting is easily accomplished for low-order systems, it is practically impossible when there are even ten parameters if there are no sufficient reasons for limiting the region of parameter change from the very beginning, i.e., for approximately prespecifying the parameter values with appreciable accuracy. This amounts partly to the need to know in advance the nature of the solution sought.

It is remarkable in difference solutions that $\ln N(\varepsilon)$ rises linearly as $\ln 1/\varepsilon$ grows larger, and, therefore, difference methods essentially permit ε to vary widely without the time required to solve the problem growing to astronomical proportions.

* Numbers in the margin indicate pagination in the original foreign text.

It may seem that it is rather meaningless to make demands on accuracy which exceed the requirements of practice. This is in fact not so.

(1) Since we are solving a problem which is physically idealized to a certain degree, we must in principle find it possible to carry out the calculation with a degree of error less than the error introduced by the physical idealization, so that we may state the problem more precisely. /248

(2) Very often the decisions being sought have a number of singularities and subtle details, and when we do not take the latter into proper consideration we may be mistaken in the values of the functionals which are required for practical necessities. It is clear to everyone that if a solution has a singularity, then in principle a small interval in the neighborhood of the singularity is required.

(3) In some fields of technology, the impossibility of conducting experiments results in the fact that computations are a basic source of information for engineers. Therefore, the quality of the computations and the degree of error are of primary significance. In gas dynamics, we may also have the situation where computations may in principle replace experimentation.

The conclusions which may be drawn from numerical computations or approximate methods are not, however, logically substantiated until the degree of error has been theoretically calculated. In all the problems analyzed below -- and this is also true of all gas-dynamics problems -- not only has no theoretical error estimate been obtained to date, but also the convergence of difference equation solutions toward the boundary-value problem solution has not even been proven.

Computational accuracy must therefore, for the time being, be examined empirically by subdivision of the difference network intervals and subsequent study of the nature of the error. Consequently, the numerical algorithm must tolerate an appreciable decrease in network intervals.

From the above, it is clear what fundamental significance may be assigned to the nature of the increase in function $N(\varepsilon)$ as ε decreases. It is our opinion that, with proper formulation of the numerical algorithm, difference methods give an order of increase $N(\varepsilon)$ which is close to optimum, if the comparison is made for a certain class of problem.

Numerical algorithms for solving gas-dynamics problems ultimately reduce to algorithms for solving several boundary-value problems for a system of quasi-linear equations. Successful formulation of the numerical algorithm therefore requires a profound understanding of the nature and character of the corresponding boundary-value problems.

It is a matter of general knowledge that various types of equations in supersonic and subsonic flow regions lead to essentially different mathematical statements of the problems. In the case of supersonic flow, where the equations are of the hyperbolic type, we may conduct the calculation in succession from some initial surface downstream. This appreciably simplifies the matter. In

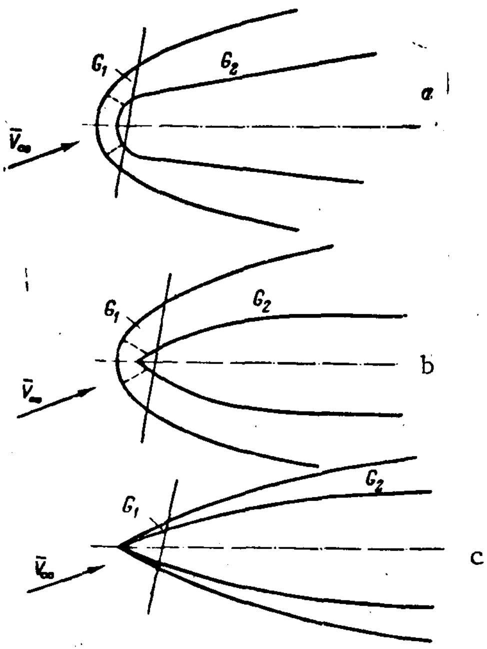

Figure 1

the case of equations of the mixed type, the problem must be solved in the whole region at once. In our opinion, the adjustment method is the best for such problems. The essence of this method consists of the fact that a stationary or self-similar solution is found as the limit of a non-stationary or non-self-similar solution.

In the numerical solution process, we have always ascertained the fact that the non-stationary solution converges to a stationary solution, or a non-self-similar solution converges to a self-similar solution. Moreover, in the numerical solution it was found that the limiting equations are strongly attracting, and this last circumstance plays a decisive role in the employment of the adjustment method.

Let us speak about this in somewhat greater detail. In the difference method of solving boundary-value problems for non-linear equations, we ultimately reduce the matter to solution of a system of non-linear equations of an order which is often rather high. This system of algebraic equations clearly has no unique solution. Actually, we encounter a problem of this sort in any iteration method of solving the problem. And it is extraordinarily important that the solution we seek be strongly con- /249
vergent. There are no general methods for ascertaining this fact in the formally-written, numerical algorithm. We therefore believe that the way out may be found by turning to the physical meaning of the problem.

The adjustment method may be treated as a certain iteration method having a physical meaning, and it may be assumed that the existence of a limit in the physical problem guarantees convergence of these iterations. This hypothesis is brilliantly confirmed. The empirically ascertained property that stationary solutions are strongly convergent ensures convergence to the needed solution.

Application of iteration methods permits reduction of the solution of non-linear algebraic equations to the solution of linear equations for which the well-developed method of matrix fitting or a combination of the fitting method plus iterations may be used.

Let us take a look at the problem of stationary flow about an arbitrary smooth body by a supersonic stream of non-viscous non-thermoconductive gas. We will regard the body as stationary, and the stream far in front of the body as uniform. In supersonic flow about a body, it is known that the region of the unperturbed uniform stream in front of the body will be separated from the region of perturbed flow by the bow shock wave.

We will restrict ourselves to studying flow in the region included between

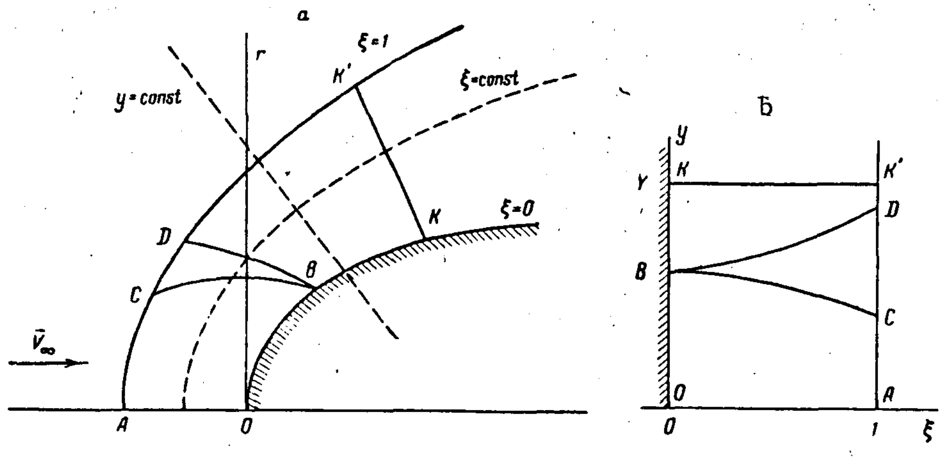

Figure 2

the bow wave and the body right up to the next discontinuity, if one occurs. In supersonic flow, perturbations may be transmitted only downstream, if they are not so large as to completely change the qualitative flow pattern. Therefore, the flow in the region in question may be found without solving the problem as a whole. In a number of cases, this is sufficient for practical purposes. Depending on the shape of the forward part of the body and the Mach number of the unperturbed flow, the forward part may be either detached (Figure 1a, b) or attached (Figure 1c). In cases a and b, a subsonic region is generated behind the bow wave (the dashed line on the diagram indicates the boundary of this region). In case c, the flow has a singularity at the apex of the body.

Because of this nature of the flow, it is convenient to divide the problem of determining the stream in the region between the shock wave and the body into two parts. Let us divide this region by some surface Π into the regions G_1 and G_2, in such a way that at all points of Π the flow velocities will be supersonic, and surface Π will be of the three-dimensional type. Determination of flow in region G_1 may be reduced to solving a boundary-value problem which differs for cases a, b, and c. After solving this problem, let us find the values of velocity $\bar{V}$, pressure p, and density ρ on surface Π, after which the flow in region G_2 may be found.

We will dwell first on the method of determining flow in region G_1 in the case when the body has a blunt nose (Figure 1a), and between the shock wave and the body a mixed flow occurs. Determination of flow in G_1 may be reduced to solving a boundary-value problem for a system of quasi-linear equations of the mixed type with three independent variables. The region in which the solution is sought is limited, in the general case, by the body surface, the shock wave, and the limiting characteristic surface, while the shape of the two latter surfaces is not known in advance. /250

Limiting conditions are set only on the part of the region boundary in which the solution is sought. In contrast to equations of the hyperbolic type, the effect of each point is here extended to the whole region; this must also occur for stable difference system. These two conditions cause certain difficulties in the numerical solution of the problem, especially in the three-dimensional case where it is required that a very considerable number of non-linear difference equations be solved. The situation as a rule is appreciably simplified in the case where the bow section of the body is spherical in shape. Specifically, if the line of intersection of the body surface with the limiting characteristic surface delimiting the region of influence is entirely situated on the spherical part in this case, the flow in the mixed region will be axisymmetrical relative to the direction of the velocity vector of the incident stream. For blunt cones and other bodies with spherical blunt forebodies, the axial symmetry in the nose region may be maintained up to substantial angles of attack, even exceeding half the aperture angle of the cone. Flow in the supersonic region will, moreover, of course be essentially three-dimensional. With the above in mind, we will study the calculation method only for axisymmetrical flow around a blunt body with arbitrary bluntness.

Let us examine a body of rotation situated in a supersonic gas stream whose velocity of $\overline{V}_\infty$ is directed along the body's axis of symmetry. Let p_∞ and ρ_∞ be pressure and density in the unperturbed flow. We will consider the unperturbed gas to be ideal with an adiabatic exponent of $k_\infty = 1.4$. Let z, r, ϑ be cylindrical coordinates with axis z along the axis of the streamlined body. By virtue of the problem symmetry, the bow wave will also be a surface of rotation, and it is sufficient to examine the flow ϑ = const. in any one meridian plane (Figure 2a).

Let u, v be components of velocity vector $\overline{V}$ along axes z, r; let p be pressure; and let ρ be density. The boundary value problem for these quantities as functions of z, r is formulated as follows:

(1) Desired functions u, v, p, ρ in the region between the shock wave and the body satisfy equations

$$\begin{gathered}\frac{du}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial z}=0;\\ \frac{dv}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0;\\ \frac{d\rho}{dt}+\rho\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}+\frac{v}{r}\right)=0;\\ \frac{dp}{dt}+\rho c^2\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}+\frac{v}{r}\right)=0.\end{gathered} \tag{1}$$

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Here

$$\frac{d}{dt}=u\frac{\partial}{\partial z}+v\frac{\partial}{\partial z},$$

$c^2 = c^2\,(p, \rho)$ is the square of the speed of sound.

(2) The desired functions satisfy the following conditions:

(a) On the body surface

$$V_n = -u \sin\varphi + v \cos\varphi = 0, \tag{2}$$

where V_n is the normal velocity component on the body and ϕ is the angle between axis z and a tangent to the body;

(b) On the shock wave:

$$\begin{gathered}\rho V_\nu = \rho_\infty V_{\nu_\infty};\\ V_\tau = V_{\tau\infty};\\ \rho V_\nu^2 + p = \rho_\infty V_{\nu_\infty}^2 + p_\infty;\\ h(p, \rho) + \frac{V_\nu^2}{2} = h_\infty + \frac{V_{\nu_\infty}^2}{2}.\end{gathered} \tag{3}$$

Here V_ν and V_τ are normal and tangential components of velocity on the shock wave; h (p,ρ) is enthalpy per unit of gas mass; $V_\nu = -u \sin\psi + v \cos\psi$; $V_\tau = u \cos\psi + v \sin\psi$, ψ is the angle between axis z and the tangent to the shock wave.

(c) On the axis of symmetry v = 0.

It should be borne in mind that the value of ϕ is known at any point of the body surface, while the value of ψ is not known in advance, and may be determined at the same time as the wave shape.

The above-formulated conditions are not at all formally connected with the presence of the sonic line and the limiting characteristic inside the region. However, in solving the problem, their position may be taken into account in some way or other, depending on the method of solution. Thus, if -- in solving the problem -- we remain within the limits of stationary equations, it is almost unavoidably necessary to trace the exact position of sonic line BC and limiting characteristic BD. In solving the problem by the adjustment method, it is unnecessary to determine the exact position of the limiting characteristic, but we must make sure that its whole course is within the region in which the solution is sought.

The physical fact underlying the adjustment method is that under real conditions, stationary flow around a body always occurs as a limit to non-stationary flow during sufficiently protracted motion of the body at a constant speed and with constant external medium parameters. It is, moreover, assumed that this limiting flow does not depend on the manner in which entrance into the stationary regime occurred. It is, therefore, to be expected that under constant boundary conditions on the body and at infinity the solution of the non-stationary flow problem will tend toward the solution of the stationary problem as time t tends toward infinity, regardless of what the original flow was. Since equations of non-stationary gas flow are always hyperbolic, the matter reduces /252
to solving a boundary-value problem for a hyperbolic system of differential equations. A mathematical formulation of the non-stationary problem is immediately obtained from the above-formulated stationary problem. In the differential

equations, however, the total derivative must be written with due regard for the explicit dependence of the functions on time $d/dt = \partial/\partial t + u \cdot \partial/\partial z + v \cdot \partial/\partial r$.

The boundary conditions on the body and on the axis remain unchanged. Under boundary conditions on the wave $V_\nu - D$ must be written, instead of V_ν, where D is shock wave velocity.

When t = 0, the position of the shock wave and the distribution of all functions in the region between wave and body are prescribed. For convenience of the numerical solution, this region should also be bounded on the side of supersonic flow. For this purpose, it is sufficient to draw line KK' lying outside the region of influence, such that the cylindrical surface constructed on it (parallel to axis t) is of the three-dimensional type. This ensures correct formulation of the problem in the absence of any boundary conditions on KK' (Figure 1a).

Let us go on to note that, if the problem is regarded in its strict mathematical formulation, the values of the hydrodynamic quantities on the body will in general not be close to those which should prevail in stationary flow. This follows from the fact that -- because the normal component of velocity on the body surface equals zero -- the change in entropy on the wave does not reach the body in a finite time interval, and entropy on the body surface keeps the same value which it had in the initial data. Strictly speaking, therefore, the solution to the non-stationary problem tends non-uniformly to stationary flow as $t \to \infty$.

When the problems are being solved by the adjustment method, this effect is parasitic, and the difference system must be so constructed that the correct entropy value is established on the body. This may be readily done, since entropy perturbations are transmitted from wave to body in the difference equations in a finite number of steps in time.

Let us describe the algorithm for numerical solution of the problem. Let us introduce a new system of curvilinear coordinates ξ, y, t, linked to the wave form in such a way that in the new coordinates the region between the body and wave will have fixed boundaries. For this purpose let us set

$$z = \mathfrak{z}(y) - [G(y) + \xi F(y, t)]\left(1 - \frac{y^2}{2}\right); \qquad r = [G(y) + \xi F(y, t)]\, y. \tag{4}$$

In the new coordinates, the body equation is $\xi = 0$, while the wave equation is $\xi = 1$. The axis of symmetry corresponds to line y = 0, and line KK' to line y = Y (Figure 2b). Functions $\mathfrak{z}(y)$ and G(y) determine the shape of the body and are given. Function F(y,t) determines the wave form and is found together with u, v, p, ρ, in the process of the solution. The quantities ψ and D which enter into the boundary conditions are expressed by $\partial F/\partial y$ and $\partial F/\partial t$ by means of elementary formulas.

In this way, the conditions on the wave contain equations for determining function F(y, t). It is convenient to use matrix notation to transform the equation; then system (1) will be written as /253

$$\frac{\partial X}{\partial t} + \mathfrak{A}\frac{\partial X}{\partial z} + \mathfrak{B}\frac{\partial X}{\partial r} + \Gamma = 0, \tag{5}$$

where

$$X = \begin{Bmatrix} u \\ v \\ p \\ \rho \end{Bmatrix}; \quad \Gamma = \frac{1}{r}\begin{Bmatrix} 0 \\ 0 \\ \rho c^2 v \\ \rho v \end{Bmatrix};$$

$$\mathfrak{A} = \begin{Bmatrix} u & 0 & \rho^{-1} & 0 \\ 0 & u & 0 & 0 \\ \rho c^2 & 0 & u & 0 \\ \rho & 0 & 0 & u \end{Bmatrix}. \qquad \mathfrak{B} = \begin{Bmatrix} v & 0 & 0 & 0 \\ 0 & v & \rho^{-1} & 0 \\ 0 & \rho c^2 & v & 0 \\ 0 & \rho & 0 & v \end{Bmatrix}.$$

After transformation to new variables, equations (5) assume the form

$$\begin{gathered} \frac{\partial X}{\partial t} + A\frac{\partial X}{\partial \xi} + B\frac{\partial X}{\partial y} + \Gamma = 0, \\ A = E\xi_t + \mathfrak{A}\xi_z + \mathfrak{B}\xi_r, \\ B = Ey_t + \mathfrak{A}y_z + \mathfrak{B}y_r. \end{gathered} \tag{6}$$

The derivatives of ξ, y with respect to z, r, t are computed by the ordinary rules and in the last analysis are explicitly expressed by ξ, y, functions $\mathfrak{z}$, G,F, and their derivatives. The boundary conditions on the wave remain as before; those on the body are written in the form

$$z_y v - r_y u = 0. \tag{7}$$

We should note that the described change of variables is not at all the only one possible, and is given here only as an example. In every case, the selection of a particular change of variables is stipulated by the demands of the specific problem.

Let us introduce into space ξ, y, t the rectangular network with intervals $\Delta t = r$, $\Delta\xi = h_1 = 1/M$, $\Delta y = h_2 = Y/L$

$$t^n = n\tau;\ \xi_m = mh_1;\ y_l = lh_2;\ f(\xi_m, y_l;\ t^n) = f^n_{m,l}:\ \varkappa_1 = \tau/h_1;\ \varkappa_2 = \tau/h_2.$$

We will write the difference system for the formulated problem in similar fashion as in (Ref. 1) for the case of three-dimensional stationary flow. The role of variable x is played in our case by t, and that of variable ϑ by variable y. In the notation of (Ref. 1), the iterative difference system looks like

$$\begin{gathered} (S+I)\left\{X^{n+(q+1)}_{m,l} - \left[I + \frac{\sigma\varkappa_2}{4}(T - 2I + T^{-1})\right]X^n_{m,l}\right\} + \\ + 2\varkappa_1 A^{n+\left(\frac{q}{2}\right)}_{m+\frac{1}{2},l}(S - I)(\alpha X^{n+(q+1)}_{m,l} + \beta X^n_{m,l}) + \\ + \frac{\varkappa_2}{2}B^{n+\left(\frac{q}{2}\right)}_{m+\frac{1}{2},l}(S + I)(T - T^{-1})(\alpha X^{n+(q)}_{m,l} + \beta X^n_{m,l}) + 2\tau\Gamma^{n+\left(\frac{q}{2}\right)}_{m+\frac{1}{2},l} = 0. \end{gathered} \tag{8}$$

Here S and T are shift operators per unit with respect to m and l respectively; I is the identity operator; $X_{m,l}^{n+(q)}$ is the value of vector $X_{m,l}$ on the q^{th} iteration; and α, β, σ are system parameters, while $\alpha + \beta = 1$, $\alpha > 0$, $\beta > 0$. There is a slight difference between the manner in which the difference system is written in (Ref. 1) on the line KK' when $l = L$, where the derivative with respect to y is replaced by the one-sided difference relationship

$$\left(\frac{\partial f}{\partial y}\right)_{m+\frac{1}{2},\,L}^{n+(q)} \cong \frac{f_{m+\frac{1}{2},\,L}^{n+(q)} - f_{m+\frac{1}{2},\,L-1}^{n+(q)}}{h_2}.$$

No boundary conditions are posed when $l = L$. The system of algebraic equations for $X_{m,l}^{n+(q+1)}$ is solved separately for each l by the fitting method described in (Ref. 1).

The method of solving equations on the wave is, of course, different from that described in (Ref. 1), where the stationary wave is examined. In the non-stationary case, it proves to be more convenient not to reduce the problem to the solution of a cubic equation, but to make direct use of Newton's method for finding the solution of the entire non-linear system. After determining $F_{t,l}^{n+(q+1)}$, the value of $F_{l}^{n+(q+1)}$ is found by integrating the equation $\partial F/\partial t = F_t$ with respect to t, e.g., by formula /254

$$F_l^{(n+(q+1)} = F_l^n + \tau(\alpha F_{t,l}^{n+(q+1)} + \beta F_{t,l}^n). \qquad (9)$$

Let us note, however, that in some cases, in calculating with this formula, the values of F_l prove to fluctuate drastically from point to point, and the graph of function F with respect to y assumes a saw-toothed appearance. In other words, the parasitic term $\delta F_l = c\,(-1)^l$ is added to the exact values of F_l. Without pausing to study this question, let us merely remark that coefficient c (generally dependent on l) is of order $O(h_2)$. But the approximation error of the system is of the second order with respect to h_2; therefore, addition δF_l may considerably exceed it and reduce the computational accuracy. In order to eliminate this defect, when $F_{t,l}^{n+(q+1)}$ is calculated, the presence of the parasitic term in $F_l^{n+(q)}$ must be taken into consideration, and it must somehow be neutralized. After the system for the wave is solved, there is a reverse and all values of $X_{m,l}^{n+(q+1)}$ are calculated for m = M-1, ..., 1,0. The iterative compuation is finished when the systems are solved for all values of l. Finally, after completion of the prescribed number of iterations Q, we finish computing the n + 1 layer, assuming that $X_{m,l}^{n+1} = X_{m,l}^{n+(Q)}$, $F_j^{n+1} = F_l^{n+(Q)}$.

The computation of $X_{m,l}^n$ continues until inequality $\max\limits_{m,l} \| X_{m,l}^{n+n_0} - X_{m,l}^n \| < \varepsilon$ is fulfilled, where n_0 and ε are given values.

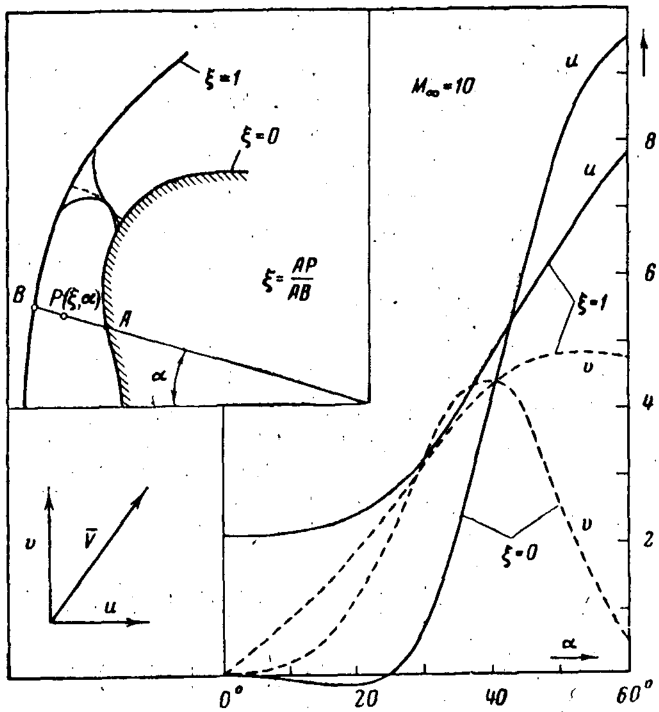

Figure 3

Figure 3 illustrates the results derived from computing flow about the nose of the body with a sector of negative curvature when M_∞ = 10. It is interesting to note that the arrangement of the characteristics here is the same as for a sphere for appreciably smaller M_∞. /255

Proceeding now to compute flow in region G_2,we would like to remark that the numerical algorithm for this was first proposed in (Ref. 2). (Ref. 1), as already stated above, presents it in detail and presents the needed studies to substantiate it. In the following, we will hold to the notation adopted there.

Let us pause on several aspects of the problem of supersonic flow of an ideal gas past smooth bodies, and specifically on computing the flow past long smooth bodies and three-dimensional flow past cones.

First of all, we will say a few words about the connection between the computations in G_1 and G_2. Into region G_2, as into G_1, we introduce the cylindrical coordinates z, r, ϑ. Since axis z in region G_2 must always have its course inside the body,z, r, ϑ will, if need be, differ from the coordinates in G_1 (let us designate them by z', r', ϑ'. Therefore, in putting the automatic transition from G_1 to G_2 into effect, a number of technical difficulties arise which involve the need to pass from one three-dimensional network to another. Let us analyze the case where the body has a spherical forebody and the flow in the forward section is axisymmetrical. As surface Π, let us take a plane perpendicular to the z axis. If the line of intersection of Π and the body lies on the spherical part of the bluntness, the flow in plane Π will be found from a computation of the axisymmetrical flow around a sphere. To calculate flow in region G_2, we must know $\bar{v}$, p, ρ on lines ϑ = const, lying in plane Π. The image of each such line in plane z', r' gives a certain curve (hyperbola), $\alpha(z', r')$ = const which will intersect the body and the shock wave.

Line ξ = const in plane Π will change into curve $\beta(z', r')$ = const in plane z', r'. Thus, region $0 \leqq \xi \leqq 1$, $0 \leq \vartheta \leq \pi$ in plane Π will be homeomorphic to the region in plane z', r' delimited by the shock wave, the body, and the two hyperbolae $\alpha(z', r')$ = const corresponding to angle values $\vartheta = 0$ and $\vartheta = \pi$. After determining flow in the mixed region, we will derive the original data in the supersonic part of the flow, and consequently we can find $\bar{V}$, p, and ρ downstream by solving the mixed problem for the hyperbolic system. It is easy to introduce new variables in plane z', r' so that the coordinate lines will be one of the rays y = const (lying in the supersonic region) and the lines α = const and β = const. By solving the mixed problem in these variables by the same algorithm which is employed in region G_2, but for two independent variables, we automatically obtain $\overline{V}$, p, and ρ in Π. Here the role of "time" will be played by coordinate z, and we will derive the values of the desired functions at the points of the difference network which is erected in region G_2. Use of this procedure disposes with the need for complex interpolation with respect to three independent variables, and makes it possible to proceed automatically from computing region G_1 to computing region G_2 when the problem is solved by a computer.

After obtaining the solution in plane Π, we will find the values of the downstream hydrodynamic quantities by employing the numerical algorithm described in (Ref. 1). However, if the body is sufficiently long, a number of difficulties will arise in the course of this solution.

Let us explain the nature of these difficulties by the example of axisymmetrical flow past a blunt cone and a blunt cylinder. At great distances from the nose, on the body surface there appears a region with marked entropy and density gradients -- the turbulent layer. In z, ξ coordinates, the thickness of the turbulent layer tends toward zero when $z \to \infty$, but the body's density gradient tends toward ∞. Therefore, with large z values the order to which the differential equations are approximated by the difference equations becomes smaller, and, consequently, the numerical solution error also increases, unless /256
we provide for specific refinement of the mesh interval along the coordinate ξ when the ξ values are small. The law of mesh interval refinement along coordinate ξ is not universally valid, but depends on the solution. Technically, it is more convenient to replace the non-uniform mesh with respect to ξ by new variables $x, \eta, z = x, \eta = \eta\ (\xi, z)$ calculated in such a way that the uniform interval along coordinate η will give the requisite mesh interval refinement along coordinate ξ at small values of ξ.

We will present a few computational examples. Figures 4 and 5 give graphs

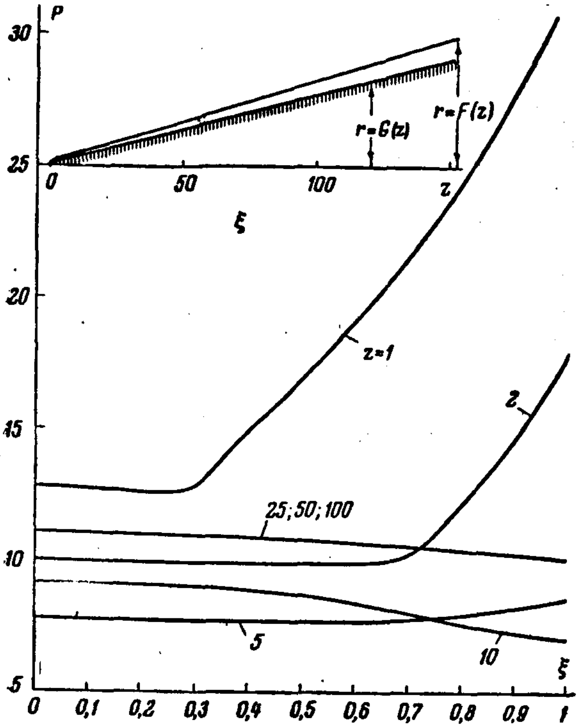

Figure 4

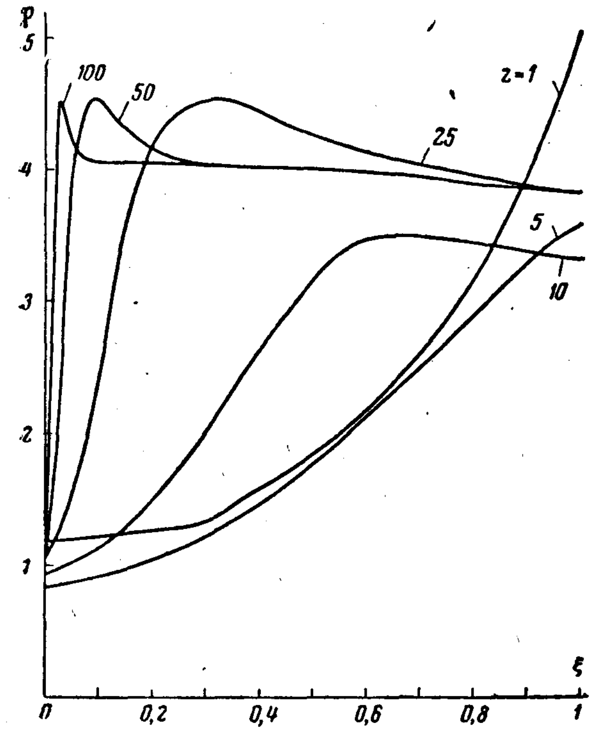

Figure 5

of variables p and ρ obtained when a perfect gas flows past a blunt cone with semi-aperture angle of 15^{0} when M_{∞} = 10 and k = 1.4. All the linear dimensions are given in terms of the radius of the spherical bluntness; pressure is stated in terms of pressure p_{∞} of unperturbed flow, and density is given in terms of density ρ_{∞} of unperturbed flow.

Figure 4 gives the relationship of p (z, ξ) as a function of ξ with fixed values of z. Coordinate z is measured from the nose of the body. The upper part of the figure shows the cross-section of the body with a meridian surface and the shock wave trace.

The number on each curve indicates the value of coordinate z. It is apparent on the drawing that when $z \geq 25$ the pressure hardly depends on z (the difference between pressures in these cross-sections is fractions of one percent) and coincides with its limiting value when $z \to \infty$, i.e., with pressure on a sharp cone. The corresponding picture for density differs sharply from the preceding. Figure 5 shows relationship ρ (z, ξ) as a function of ξ with fixed z values. The values of coordinate z are indicated on each curve. It is apparent that even at a distance of 100 gradations from the body nose, the density is close to its limiting value only when $\xi > 0.1$. On the body, however, a region forms with sharp ρ gradients (turbulent layer). Density is a nonmonotonic function of ξ within this layer.

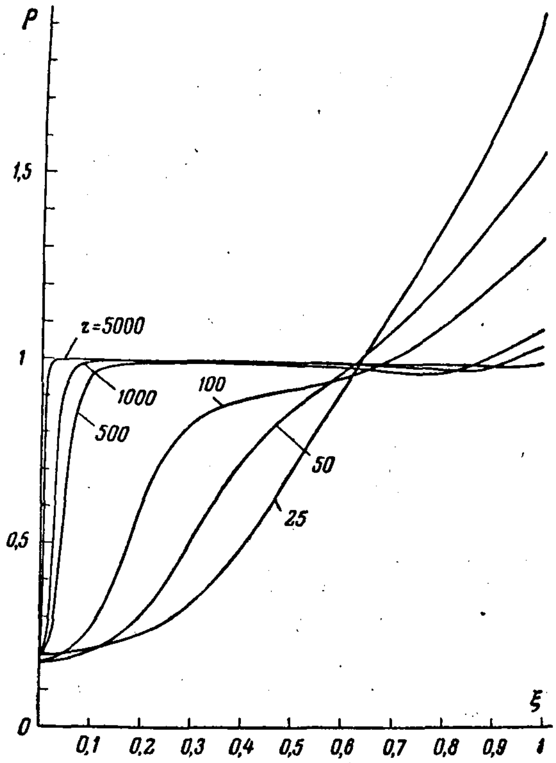

Figure 6

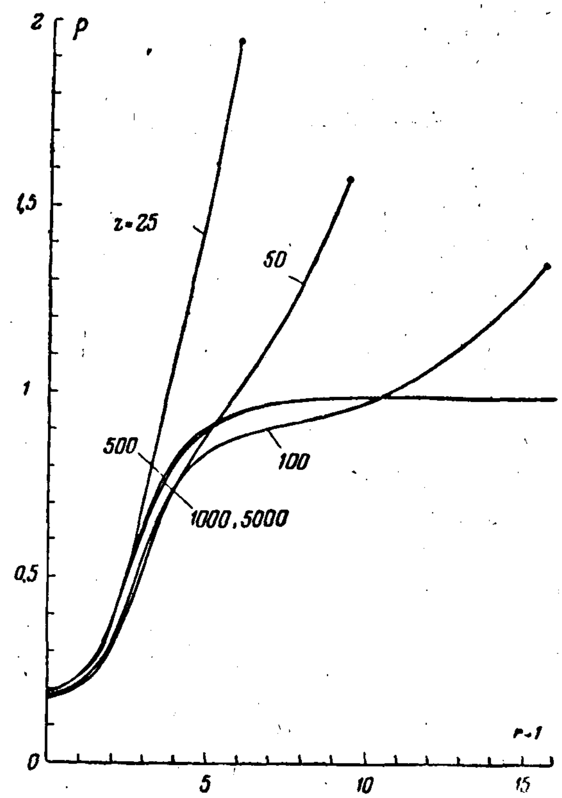

Figure 7

In addition, Figure 5 clearly shows formation of the Gibbs phenomenon. On the ρ graph, there is a characteristic "hump" whose height does not depend on z, and this "hump" shifts merely to the left as z gets larger. Let us note that, depending on the shape of the body in its nose, there may even be several of these humps, or they may be altogether lacking, as in flow past a blunt cylinder.

For purposes of comparison, Figure 6 shows the relationship ρ (z, ξ) in the case of flow past a cylinder with spherical bluntness with the same number of $M_\infty = 10$.

Figure 7 shows the relationship ρ (z,r - 1) (cylinder radius is unity) as a function of r - 1 with fixed z. Here it is clearly evident that even at z = 500 the profile of ρ in the turbulent layer is only very weakly dependent on z, while at z = 1000 and z = 5000 the ρ profiles coincide with a great number of indices and are congruent with the asymptotic relationship ρ (r - 1). Figure 6 shows that with values of ξ close to unity -- i.e., in the neighborhood of the shock wave -- a region of flow overexpansion is formed at large values of z. The formation of the N-wave may be traced in a more detailed picture.

In three-dimensional flow, the indicated effects appear still more sharply, especially at large angles of attack.

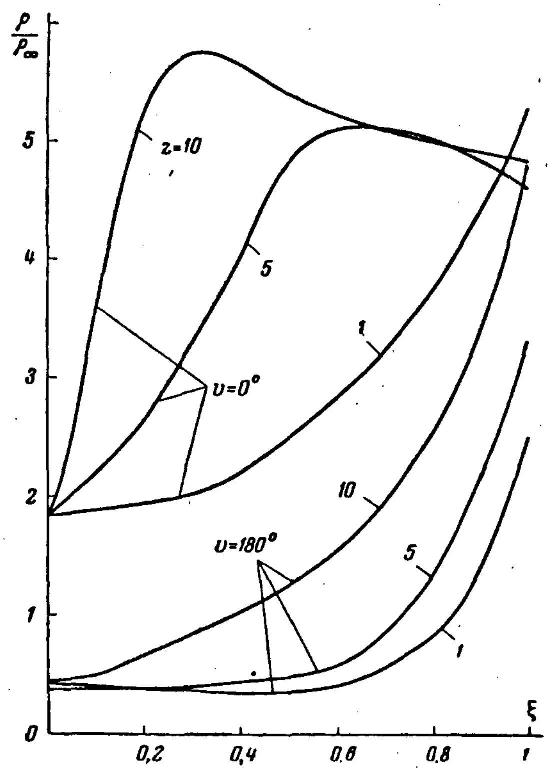

Figure 8

Figure 8 gives density distribution in flow past a blunt cone with semi-aperture angle $\beta = 10^\circ$ at an angle of attack $\alpha = 15^\circ$ with $M_\infty = 10$. The flow pattern on the upstream side differs greatly from that on the downstream side. Great density gradients are formed on the body surface even at comparatively small z values, and therefore a non-uniform mesh must be used in practice with z = 10.

Figure 9 gives values of circular velocity component w on the body surface in several cross-sections for $\beta = 10^\circ$, $M_\infty = 10$, and α = 5, 10, and 15°.

Figure 10 gives the shapes of shock waves for $\beta = 10^\circ$ and $\alpha = 15^\circ$ with $M_\infty = 4$ and $M_\infty = 10$. /259

The problem of three-dimensional flow past long smooth bodies with the conic flows computed by the adjustment method must be regarded in a different aspect. The adjustment process may be followed in the calculation of the blunt cone shown in Figures 4 and 5. These figures also show the remarkable fact that pressure coincides with great accuracy with its own limiting value even at small values of coordinate z, while density rather greatly differs from its limiting value even at large z values and tends non-uniformly toward its limiting value.

In calculating flow around the cone, however, we are interested only in the limiting flow pattern, not with its complex development. Therefore, in order to exclude difficulties in calculating flow on the body, we must start with initial data which do not lead to a complex flow picture in the vicinity of the body surface. In addition, the difference system used to compute flow in the turbulent layer maintains a constant entropy value on the body with great accuracy. This results in sharp gradients of the quantity ρ. In computing conical flows we must use a difference system which possesses the opposite property -- it rapidly "forgets" the initial values of entropy on the body and transmits entropy perturbations from shock wave to body at the greatest possible speed.

In order to obtain a correct value of conic flow parameters in the whole closed region within which the variables ξ, ϑ change, we must take into account the singularities of the conic sections. A. Ferri (Ref. 3) was the first to show that the solution to the problem would be discontinuous in flow past round

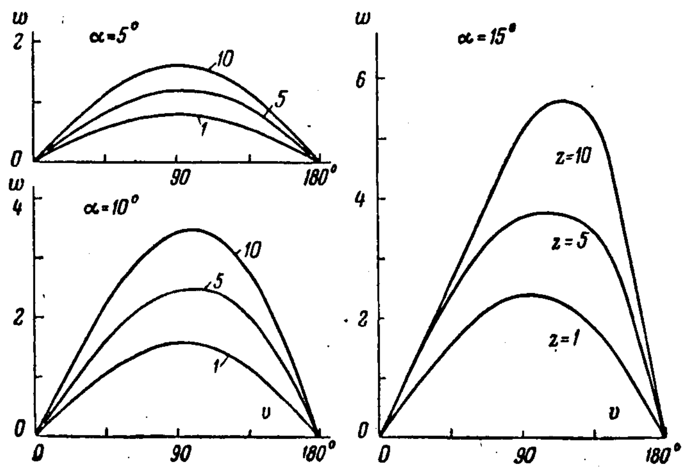

Figure 9

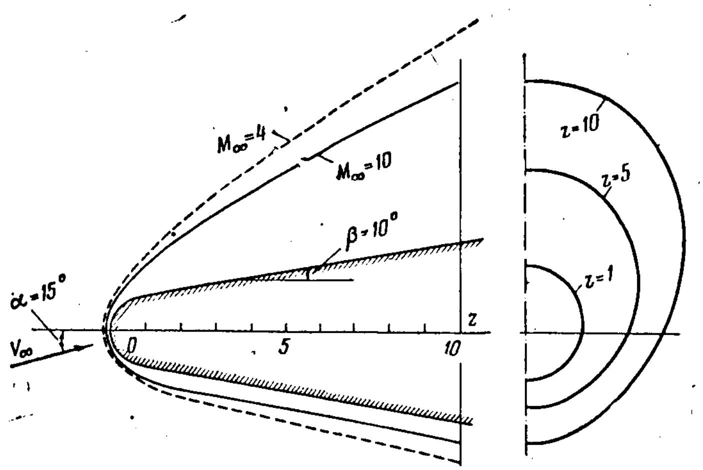

Figure 10

and elliptical cones. Entropy S necessarily has points of discontinuity in the closed region $0 \leq \xi \leq 1$, $0 \leq \vartheta \leq 2\pi$. In addition, it may be shown that on the surface of the cone

$$\frac{\partial S}{\partial \xi} = \infty, \tag{10}$$

while in the vicinity of the cone surface, a drastic change in entropy and velocity components occurs in a very narrow layer. This layer is conventionally called the turbulent layer. By itself, condition (10) causes no great complications in the numerical solution, but the narrowness of the turbulent layer leads to marked complications and requires the development of a suitable numerical algorithm. Therefore, in the solution process we must take into consideration the presence of particular entropy points and the nature of flow in the turbulent layer, in order correctly to obtain the numerical solution. Lack of space forbids us to describe the numerical algorithm with any degree /260
of accuracy. Let us merely note that in this algorithm it is essential to take into consideration the behavior of the solution in the vicinity of singular

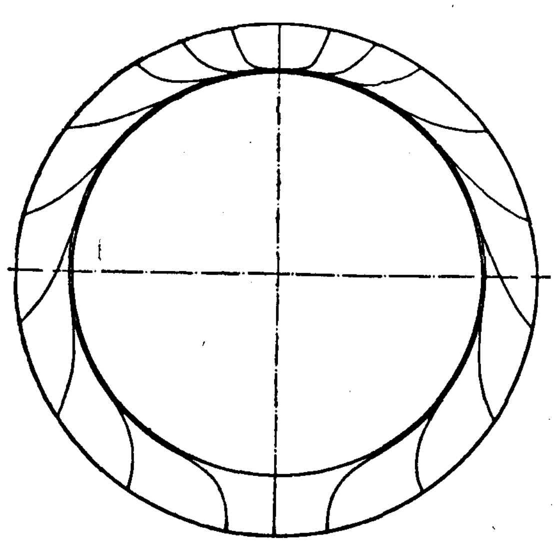

Figure 11

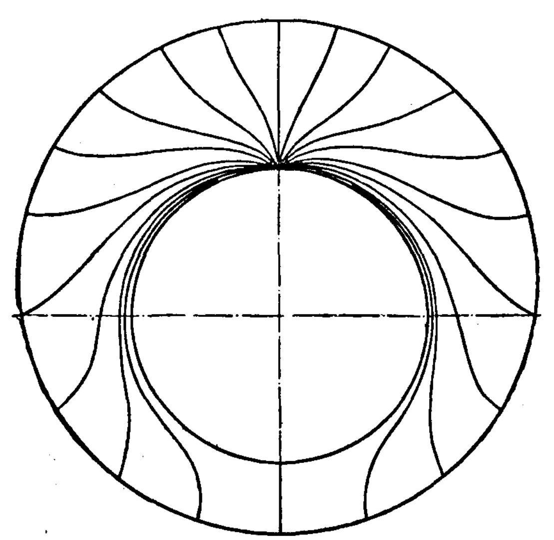

Figure 12

points, while their number, nature, singularities, and the location of the singular point may be determined in the computational process.

The location of the singular points is closely connected with the general properties of the boundary-value problem to which the conic flow problem reduces. In fact, the isoentropicity equation for conic flows has the form

$$\left(\xi_z u + \xi_r v + \frac{1}{r}\xi_\vartheta w\right)\frac{\partial S}{\partial \xi} + \frac{w}{r}\frac{\partial S}{\partial \vartheta} = 0. \qquad (11)$$

Therefore, at the points of the entropy discontinuity, it is necessary that $\xi_z u + \xi_r v + \frac{1}{r}\ \xi_\vartheta w = 0$ and $w/r = 0$.

As shown in (Ref. 1), however, it is necessary, for the boundary-value problem to be correct, that condition

$$\xi_z u + \xi_r v + \frac{1}{r}\xi_\vartheta w < 0. \qquad (12)$$

be fulfilled in the region between wave and body.

Impairment of this condition results in incorrectness of the boundary-value problem and instability in the solution to the mixed problem for the hydrodynamic equation system /261

$$A\frac{\partial X}{\partial z} + B\frac{\partial X}{\partial \xi} + C\frac{\partial X}{\partial \vartheta} + \Gamma = 0.$$

It therefore seems more probable that the singular entropy points will always lie on the surface of the cone where by virtue of the zero flow condition

$$\xi_z u + \xi_r v + \frac{1}{r}\xi_\vartheta w = 0.$$

Figures 11-13 show some of the results derived from calculating conic flow. (Ref. 1) contains systematic tables of flow past circular cones at angles of attack.

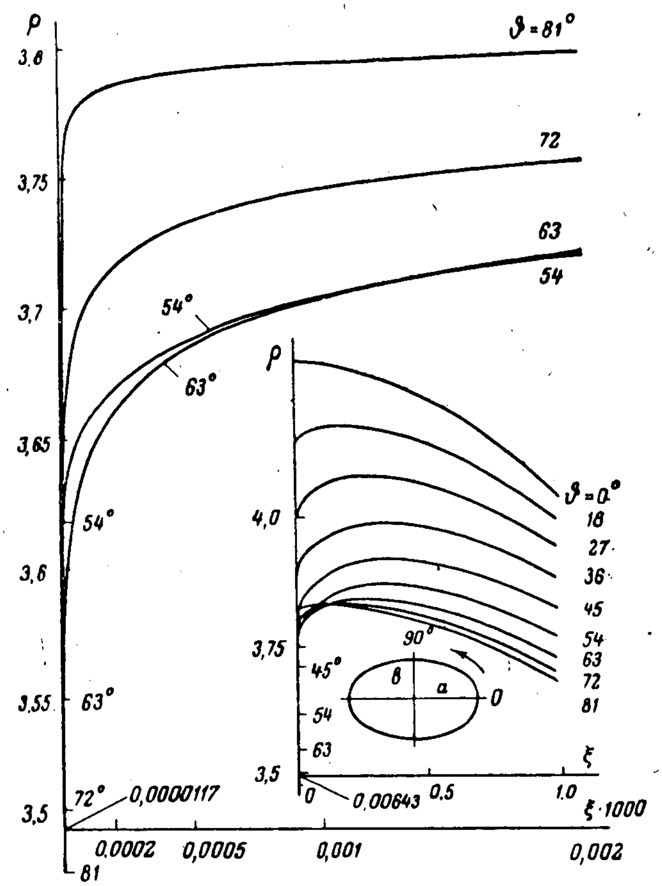

Figure 13

Figure 11 presents a shock wave trace and the lines of entropy level obtained in flow past a circular cone with a serm-aperature angle of $\beta = 25°$ ascertained at an angle of attack $\alpha = 20°$ with a perfect gas flowing past at $M_\infty = 3$. Figure 12 represents a similar pattern for $\beta = 35°$, $\alpha = 10°$, and $M_\infty = 5$. This case is characterized by the fact that a reversed position of the shock wave takes place. On the upstream side, the wave is less pressed to the body than on the downstream side. /262

Figure 13 gives the relationship of density within the turbulent layer for an elliptical cone with semi-axis ratio a/b = 1.333. Angle of attack is $\alpha = 0$ and $M_\infty = 5$. In the lower right corner the curves of $\rho\ (\xi, \vartheta)$, $0 \leq \xi \leq 1$ are shown as functions of ξ with fixed angle ϑ. Angle ϑ is measured counter-clockwise from the major semi-axis. The larger graph depicts the same functions $\rho\ (\xi, \vartheta)$ with ϑ = const, but the scale on the ξ axis is enlarged 1000 times, but only 5 times on the ρ axis. On the ρ axis points are arranged giving the angle in degrees. Each point gives the density value on the cone surface for the corresponding line ϑ = const.

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